

Properties of Addition and Multiplication of Real Numbers

Let R be the set of real numbers.

Let $a, b,$ and $c \in R$. $a, b,$ and c are real numbers.

	Addition	Multiplication	Sets These are examples.
Closure:	$a + b \in R$ Words: Add (or multiply) two real numbers and we get a real number.	$a \cdot b \in R$	$A \cup B$ is a set.
Commutative:	$a + b = b + a$	$a \cdot b = b \cdot a$	$A \cup B = B \cup A$ $A \times B \neq B \times A$
Associative :	$(a + b) + c = a + (b + c)$	$(ab) \cdot c = a \cdot (bc).$	
Identity :	$\exists 0 \in R \ni a + 0 = a$ Words: There exists a number such that any number added (multiplied) to this number gives the number.	$\exists 1 \in R \ni a \cdot 1 = a$	$A \cup \emptyset = A$
Inverse :	$\forall a \exists (-a) \ni a + (-a) = 0$	$\forall a \neq 0 \exists \frac{1}{a} \ni a \cdot \frac{1}{a} = 1$	
	Words: For any number there is another number called the inverse such that the number and the inverse is the identity number.		
Distributive :	$a \cdot (b + c) = a \cdot b + a \cdot c$ Note that this is for both addition and multiplication.		

Example how we can use the inverse property to prove that $-(-a) = a$:

Let us take the inverse of $-a$. This inverse is a number by the inverse property.

Using the inverse property on $-a$, we have

$$-a + (-(-a)) = 0, \text{ or}$$

$$-(-a) = a, \text{ proving it.}$$

Complex numbers.

We know the world is 3 dimensional, as we need 3 numbers to specify the location of a point in space. We tend to think of numbers as one-dimensional. Actually, numbers are two-dimensional.

Consider the equation $x^2 + 1 = 0$. There is no real number that satisfies this.

Mathematicians created a number i , where $i^2 = -1$, and called it an imaginary number.

We can write any number as $a + ib$, called a complex number. Here a is the real part, and b is the imaginary part. A complex number is an ordered pair. Note that i is a number, not a variable.

There is a theorem that states that all solutions of any algebraic equations can be written in terms of complex numbers. We do not need a third dimension for numbers.

E.g, suppose we had an equation like $x^3 = -8$. Let us guess the solution is $x = j$, where j is a third dimension imaginary number. Well, we can find

j . We have $j^3 = -8$. The solution is

$$j = (1 + i\sqrt{3}).$$

Proof:

If we expand $(a + b)^3$ we have $a^3 + 3a^2b + 3ab^2 + b^3$.

$$j^3 = 1 + 3i\sqrt{3} - 3\cdot 3 - 3i\sqrt{3} = -8$$

We see that we do not need a new type of complex number j , as i is good enough.

Note that $\sqrt[3]{-8} \neq -2$.

Another interesting thing:

$$(a + ib)(a - ib) = a^2 + b^2. \text{ Just expand.}$$

The distance from a point in the complex plane to the origin is called the **absolute value**.

$$|a + ib| = \sqrt{a^2 + b^2}.$$

Here is something that we will not prove:

$$\cos \theta + i \sin \theta = e^{i\theta}$$

This equation is one of the basic equations needed for engineering, such as power generation and transmission.

If $\theta = \pi$, then $e^{i\pi} = -1$. All the amazing numbers of mathematics in one equation: $e, i, \pi, -1$.