

We have shown that $dz = xdy + ydx$ is an exact differential, as $\int_a^b xdy + ydx$ depends only on the end points a and b , and not on the path. The proof is very simple:

$$\int_a^b xdy + ydx = \int_a^b dxy = xy|_b - xy|_a$$

To understand this better, it is helpful to take a simple function and perform the integration. Let c be an arbitrary point $a < c < b$. Let $y(x)$ be a straight line between a and c , and a line between c and b .

We will show that the integral does not depend on the point c .

Let y be a line between points a and b .

$$y = Ax + B$$

$$y_a = Ax_a + B$$

$$y_b = Ax_b + B$$

Solving:

$$y = \frac{(y_b - y_a)}{(x_b - x_a)}x + y_a - \frac{(y_b - y_a)}{(x_b - x_a)}x_a$$

Integrating and collecting terms we get

$$\int_a^b ydx = (x_b - x_a)(y_b + y_a) / 2$$

Now do the same thing writing $x(y)$, such as

$$x_a = Ay_a + B, \text{ where } A \text{ and } B \text{ are not the same as for the equation for } y_b.$$

We have the same thing, with x and y interchanged. We get:

$$\int_a^b xdy = (y_b - y_a)(x_b + x_a) / 2$$

Let us use these results to evaluate $\int_a^c + \int_c^b$. We get:

$$x_c y_c - x_a y_a + x_b y_b - x_c y_c$$

Not only do we see that the integral is independent on the point c , but also we see that

the integral $\int_a^c xdy + \int_c^b xdy$ does depend on the point c .