Formal **definition of a subset**:
$A \subseteq B$ if $\forall a \in A \rightarrow a \in B.$

The $\forall$ means that you pick a until you are done, and if $a \in A,$ etc.

Suppose $A = \varnothing$. Pick a until we are done. There is nothing to pick.

Therefore, the condition is satisfied, and $\varnothing \subseteq A$ for all sets $A.$

Definition of a proper subset:
$A \subset B$ if $A \subseteq B$ AND $A \neq B.$

Definition of set equality:
$A = B$ if $\forall a \in A \rightarrow a \in B$ AND $\forall a \in B \rightarrow a \in A.$

Set equality theorem.
If $A = B$ then A $\subseteq B$ and $B \subseteq A.$
Proof:
If $A \subseteq B$ then $\forall a \in A \ \rightarrow \ a \in B.$

If $B \subseteq A$ then $\forall a \in B \ \rightarrow \ a \in A.$
This is the definition of equality.

The definition refers to elements.
The theorem refers to subsets:

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